AN ANALYTIC APPROACH TO A GENERAL CLASS of G/G/s QUEUEING SYSTEMS

DIMITRIS BERTSIMAS

Massachusetts Institute of Technology, Cambridge, Massachusetts

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We solve the queueing system $C_k/C_m/s$, where C_k is the class of Coxian probability density functions (pdfs) of order k, which is a subset of the pdfs that have a rational Laplace transform. We formulate the model as a continuous-time, infinite-space Markov chain by generalizing the method of stages. By using a generating function technique, we solve an infinite system of partial difference equations and find closed-form expressions for the system-size, general-time, prearrival, post-departure probability distributions and the usual performance measures. In particular, we prove that the probability of n customers being in the system, when it is *saturated* is a linear combination of geometric terms. The closed-form expressions involve a solution of a system of nonlinear equations that involves only the Laplace transforms of the interarrival and service time distributions. We conjecture that this result holds for a more general model. Following these theoretical results we propose an exact algorithm for finding the system-size distribution and the system's performance measures. We examine special cases and apply this method for numerically solving the $C_2/C_2/s$ and $E_k/C_2/s$ queueing systems.

Queueing theory has been studied thoroughly throughout this century, but many problems still remain unsolved, in spite of the effort and intelligence devoted to them. Among these problems, the analysis of the GI/G/s queueing system (QS) has survived the *attacks* of many excellent mathematicians and operations researchers, obviously due to its inherent complexity.

In this paper, we generalize the method of stages, introduced early in the century by Erlang, and combine it with a generating function technique to achieve the solution of the general class of problems $C_k/C_m/s$, where C_n is the Coxian class of probability density functions (pdf) with n stages introduced by Cox (1955).

A Brief Critical Presentation of Alternative Solution Methods

When it comes to exact solutions of multiserver QSs, the more one departs from the assumption of exponentiality, the more thorny the problems become, especially if this happens for the service time pdf or, worse, for both the service and arrival-time pdf. Thus, the only solution approaches that have been established up to now as supposedly *general purpose* are

1. embedded Markov chain,
2. inclusion of supplementary variables,
3. complex variable theory,
4. the method of successive exponential stages.

For an excellent survey of computational methods for the GI/G/s queue, the interested reader can turn to Tijms (1986). Several discussions of the major computational problems arising in the numerical solution of multiserver queues with nonexponential service time also can be found in Yu (1977) and Neuts (1981). Some further remarks on the application of these solution approaches to multiserver models and on their inherent potential to produce theoretical and, especially, exact numerical results (which should be their ultimate goal), are included below.

The Embedded Markov Chain Method

In the past decade, it was mainly Neuts and his coworkers who exploited the phase-type (PH) distribution and developed the powerful formalism of the associated matrix-analytic methods and algorithmic approaches (as presented in Neuts), and managed, by *bridging* this method with the method of stages, to considerably extend its potential. Yet, for multiserver systems with nonexponential service times the relevant solutions, although providing qualitative insight, lead to major dimensionality problems. For the $C_k/C_m/s$ model, this approach requires the solution of a nonlinear matrix equation involving matrices of

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the order $\binom{s+m-1}{s}$. Ramaswami and Lucatoni (1985) extended the potential of the method by devising efficient algorithms for solving the nonlinear matrix equations involved. For the H_2 service time pdf they reported numerical results when the number of servers was 15.

The Inclusion of Supplementary Variables Method

Despite the initial expectations concerning its capabilities, the method has been used for multiserver systems in a limited number of analytic investigations and numerical implementations. In the last decade, Ishikawa (1979) tried to tackle the $G/E_m/s$ system through this method, but restricted his attention to the $G/E_3/3$ system. In parallel, Hokstad (1980) tackled the $M/C_2/s$ system, and after various simplifications presented limited results for $s = 3$. Finally, Cohen (1982) analyzed the $M/H_m/2$ system.

The Complex Variable Theory Method

This method was developed by Pollaczek (1961) and played an important role in the development of queueing theory. In recent years, the method was exploited by de Smit, who proposed a method for $G/H_m/s$ (de Smit 1983a). This approach needs deep arguments from complex variable theory, leading to a numerical solution in the $G/H_2/s$ case (de Smit 1983b), but with a high computational complexity that is proportional to s^6, to be compared with $O(s^3)$ of the present method.

The Method of Successive Exponential Stages

Erlang's method of stages has been neglected for some time, obviously because of researchers' concern about the fact that it leads to systems of equations which usually are complicated, but certainly not more formidable than those of the previous methods, and become almost intractable if one attempts to tackle them directly by various seemingly powerful techniques. For example, in the $E_k/E_m/s$ case this intractability is apparent both in many of the earlier attempts at an exact solution via multidimensional generating function techniques, and in Yu's ambitious theoretical treatise via an intricate partitioning of the system-states and an exploitation of the cyclic structure (exhibited by the corresponding transitions) through the use of polynomial matrices.

In particular, for the latter approach (see, e.g., Hillier and Lo 1971, Hillier and Yu 1981), the computations necessary for the derivation of numerical results involve the numerical expansion of such matrices, which is an enormous task even for moderate values of k, m and s. The above mentioned attitude is reflected in the comments of Kleinrock (1975, pp. 146–147). In addition to the above, the Laguerre transform method of Keilson and Sumita (1981) is worth mentioning.

Our Result

Our result generalizes the well known GI/G/1 theory to multiserver systems in a natural probabilistic way and leads to an algorithm of a relatively low order of complexity. On the other hand, in comparison to the purely numerical methods of Takahashi and Takami (1976) (which was recently specialized by Groenevelt, Van Hoorn and Tijms (1984) for the solution of the simpler models $M/H_2/s$, $M/E_{1,2}/s$, $M/E_{1,3}/s$) and Seelen (1986), the present approach offers qualitative insight by providing closed-form expressions, which apart from their computational value, are also of theoretical interest. Furthermore, our solution strategy leads to an exact waiting time analysis under FCFS, to be presented in a forthcoming paper.

We propose an $O(k^3\binom{s+m-1}{s}^3)$ algorithm for the calculation of the performance measures and the probability distributions of this QS, which for a given m is polynomial in the number of servers. To properly test the potential and reliability of this algorithm, we prepared computer programs for the numerical solution of the QSs $E_k/C_2/s$ and $C_2/C_2/s$. These exact results are in agreement with others in the literature and can be exploited for the always desirable sensitivity analysis and comparative evaluation in the continuously active areas of approximations, inequalities, bounds and stochastic order relationships for multiserver models, on all of which there is a rapidly increasing literature.

In the next section, we formulate the model as a continuous time Markov chain using the method of stages. In Section 2, we apply a generating function technique for solving the difference equations that describe the system. In this section, which is central to the analysis, we combine results of the complex variable method developed by Pollaczek (1961) and de Smit (1983a) with results of the present paper to prove what we call the *separability property*: The probability of n customers ($n \geq s$) being in the system is a linear combination of $\binom{s+m-1}{s}$ geometric terms. In Section 2.5, we examine the QSs $C_k/C_m/1$, $C_k/M/s$, $E_k/E_m/s$ and $E_k/C_2/s$ as special cases.

The derivations of closed-form expressions for the system-size probability distributions and the usual performance measures are outlined in Section 3. In the final section, we include some computational and complexity considerations.

1. FORMULATION OF THE MODEL AS A CONTINUOUS-TIME MARKOV CHAIN

We assume that both interarrival and service times are Coxian with k, respectively, m stages. This means that an arrival has to go through up to k stages. The length of stage n is exponential with a given rate λ_n. After stage n, $n = 1, 2, \ldots, k$, the interarrival time comes to an end with probability p_n, and it enters the next stage with probability $1 - p_n$. Obviously, $p_k = 1$. A similar characterization is available for the service time, except that the symbols μ_n and q_n, $n = 1, \ldots, m$ take the place of λ_n and p_n. The Laplace transform of the pdf of the interarrival times is then

$$f^*_{T_a}(\theta) = \sum_{n=1}^{k} \frac{p_n \lambda_n}{\theta + \lambda_n} \prod_{r=1}^{n-1} \frac{(1-p_r)\lambda_r}{\theta + \lambda_r}$$

where the product for $n = 1$ is defined to equal 1.

It is remarkable that even if we permit transitions from a stage with rate λ_i to a stage with rate λ_j ($j \neq i + 1$) we do not obtain a new class of distributions. We can still formulate this situation with a Coxian distribution with different transition rates. The salient feature of the class of Coxian distributions (C_n) is its high versatility based on its ability to

1. generalize well known distributions such as the exponential, the hyperexponential and all forms (that is, special, general, weighted, compound, etc.) of the Erlangian;
2. be dense in the set of all probability distributions concentrated on $(0, \infty)$ and, thus, to be able to approximate a general pdf;
3. permit coefficients of variation V_s^2 greater than 1/n.

1.1. Notation

For the steady-state we introduce the random variables

$N \triangleq$ the number of customers in the system,
$N^- \triangleq$ the number of customers seen by an arriving customer just before arrival,
$N^+ \triangleq$ the number of customers seen by a departing customer just after departure,
$R_a \triangleq$ the number of the arrival stage currently occupied by the arriving customer,
$R_j \triangleq$ the number of customers served at the jth service stage ($j = 1, 2, \ldots, m$),
$R_j^- \triangleq$ the number of customers served at the jth service stage ($j = 1, 2, \ldots, m$), just before the arrival of an entering customer,
$T_q \triangleq$ the waiting time of an arriving customer.

For simplicity of notation we introduce the vectors of random variables

$$\vec{R} \triangleq (R_1, \ldots, R_m), \quad \vec{R}^- \triangleq (R_1^-, \ldots, R_m^-)$$

and also we use the notation

$$\vec{\delta}_j \triangleq (0, \ldots, 0, 1, 0, \ldots, 0)$$

$$a(s, m) \triangleq \binom{s + m - 1}{s}$$

$$\vec{i} \triangleq (i_1, \ldots, i_m)$$

$$|\vec{i}| = s \Leftrightarrow \sum_{j=1}^{m} i_j = s.$$

With these definitions, the system can be formulated as a continuous time Markov chain with infinite state-space

$$\left\{(N, R_a, R_1, \ldots, R_m),\ N = 0, 1, \ldots,\ R_a = 1, 2, \ldots, k,\ \sum_{j=1}^{m} R_j = \min(N, s)\right\}$$

where the states with $N < s$ (that is, the states with at least one server free) and $N \geq s$ (or all servers busy) will be termed *unsaturated* and *saturated*, respectively.

We will introduce the following set of probabilities, some of which will be used in later sections.

$$P_{n,l,\vec{i}} \triangleq \Pr\{N = n, R_a = l, \vec{R} = \vec{i}\}$$

$$P^-_{n,\vec{i}} \triangleq \Pr\{N^- = n, \vec{R}^- = \vec{i}\}$$

$$P_n \triangleq \Pr\{N = n\}$$

$$P^-_n \triangleq \Pr\{N^- = n\}$$

$$P^+_n \triangleq \Pr\{N^+ = n\}.$$

We also define

$f^*_{T_a}(\theta)$, $f^*_{T_s}(\theta)$

$\triangleq$ the Laplace transform of the interarrival and service time distributions, respectively.

As usual, λ is the arrival rate, μ the service rate, and $\rho = \lambda/s\mu$ is the traffic intensity.

1.2. The Equations

After drawing the rather complicated state-transition diagram in Figure 1 (for the case $l = 2, \ldots, k$, $n \geq s$, the case $l = 1$ is similar) we write the following system

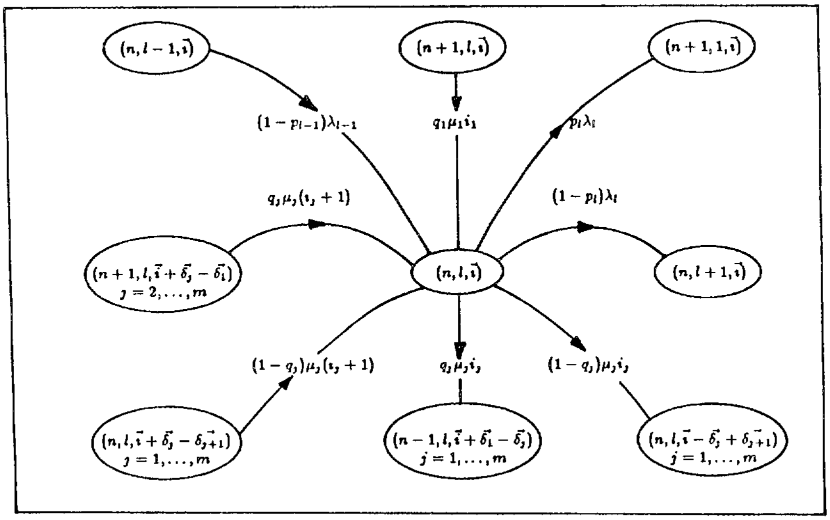

Figure 1. The state-transition diagram for $l = 2, \ldots, k$, $n \geq s$.

of equations

$\underline{n \geqslant s, \ |\vec{i}| = s}$

a. $\underline{l = 1}$

$$P_{n,1,\vec{i}}\left\{\lambda_1 + \sum_{j=1}^{m} i_j \mu_j\right\}$$

$$= \sum_{l=1}^{k} p_l \lambda_l P_{n-1,l,\vec{i}} + q_1 \mu_1 i_1 P_{n+1,1,\vec{i}}$$

$$+ \sum_{j=2}^{m} q_j \mu_j (i_j + 1) P_{n+1,1,\vec{i}+\vec{\delta}_j-\vec{\delta}_1}$$

$$+ \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) P_{n,1,\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}} \qquad (1)$$

b. $\underline{l = 2, \ldots, k}$

$$P_{n,l,\vec{i}}\left\{\lambda_l + \sum_{j=1}^{m} i_j \mu_j\right\}$$

$$= (1 - p_{l-1}) \lambda_{l-1} P_{n,l-1,\vec{i}} + q_1 \mu_1 i_1 P_{n+1,l,\vec{i}}$$

$$+ \sum_{j=2}^{m} q_j \mu_j (i_j + 1) P_{n+1,l,\vec{i}+\vec{\delta}_j-\vec{\delta}_1}$$

$$+ \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) P_{n,l,\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}} \qquad (2)$$

$\underline{n < s, \ |\vec{i}| = n}$

a. $\underline{l = 1}$

$$P_{n,1,\vec{i}}\left\{\lambda_1 + \sum_{j=1}^{m} i_j \mu_j\right\}$$

$$= \sum_{l=1}^{k} p_l \lambda_l P_{n-1,l,\vec{i}} + \sum_{j=1}^{m} q_j \mu_j (i_j + 1) P_{n+1,1,\vec{i}+\vec{\delta}_j}$$

$$+ \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) P_{n,1,\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}} \qquad (1a)$$

b. $\underline{l = 2, \ldots, k}$

$$P_{n,l,\vec{i}}\left\{\lambda_l + \sum_{j=1}^{m} i_j \mu_j\right\}$$

$$= (1 - p_{l-1}) \lambda_{l-1} P_{n,l-1,\vec{i}} + \sum_{j=1}^{m} q_j \mu_j (i_j + 1) P_{n+1,l,\vec{i}+\vec{\delta}_j}$$

$$+ \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) P_{n,l,\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}} \qquad (2a)$$

$P_{n,l,\vec{i}}$ is taken to be 0 if some component of $\vec{i}$ is negative or $|\vec{i}| \neq \min(n, s)$. Also, since $q_m = 1$ the last sum of the right-hand side of (1), (1a), (2), and (2a) can be extended to m instead of $m - 1$.

2. ANALYSIS OF EQUATIONS

2.1. Separation of Variables Technique

Initially, we consider the infinite number of equations (1) and (2) for $n \geqslant s$. We use a familiar technique from the theory of partial differential equations (see, for example, Mathews and Walker 1970, p. 226), which is less well known for partial difference equations, the separation of variables technique. This technique is presented in Mickens (1987, p. 186) and it is used in queueing theory applications in Morse (1958, p. 68), although the name separation of variables was not used in Morse. We adopt it here because it characterizes the technique well. We assume that the saturated probabilities are of the form

$$P_{n,l,\vec{i}} = D_l R_{\vec{i}} w^n \quad n \geq s.$$

We need to determine D_l, $R_{\vec{i}}$ and w. There will be several values for w, which lead to different D_l and $R_{\vec{i}}$. These values can be combined, using the initial conditions. Obviously

$$R_{\vec{i}} = 0 \quad \text{for } \sum_{j=1}^{m} i_j \neq s$$

or

$$i_j < 0 \quad \text{for some } j = 1, \ldots, m.$$

From (1) and (2) we get for $n \geq s$

$\underline{l = 1, \quad |\vec{i}| = s}$

$$D_1 R_{\vec{i}} \left\{ \lambda_1 + \sum_{j=1}^{m} i_j \mu_j \right\} = \frac{1}{w} \sum_{l=1}^{k} p_l \lambda_l D_l R_{\vec{i}} + w q_1 \mu_1 i_1 D_1 R_{\vec{i}} + w \sum_{j=2}^{m} q_j \mu_j (i_j + 1) D_1 R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_1} + \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) D_1 R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}}, \tag{3}$$

$\underline{l = 2, \ldots, k, \quad |\vec{i}| = s}$

$$D_l R_{\vec{i}} \left\{ \lambda_l + \sum_{j=1}^{m} i_j \mu_j \right\} = (1 - p_{l-1}) \lambda_{l-1} D_{l-1} R_{\vec{i}} + w q_1 \mu_1 i_1 D_l R_{\vec{i}} + w \sum_{j=2}^{m} q_j \mu_j (i_j + 1) D_l R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_1} + \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) D_l R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}}. \tag{4}$$

(4) can be written

$$\frac{D_l \lambda_l - (1 - p_{l-1}) \lambda_{l-1} D_{l-1}}{D_l} = \left[w q_1 \mu_1 i_1 R_{\vec{i}} + w \sum_{j=2}^{m} q_j \mu_j (i_j + 1) R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_1} + \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}} - R_{\vec{i}} \sum_{j=1}^{m} i_j \mu_j \right] \Big/ R_{\vec{i}}. \tag{5}$$

Since (5) holds for any combination of $l \geq 2$ and $\vec{i}$, $|\vec{i}| = s$, we apply it for the pairs $(2, \vec{i}), \ldots, (l, \vec{i}), \ldots, (k, \vec{i})$, that is, we keep $\vec{i}$ fixed and vary l. In this way we obtain

$$\frac{D_2 \lambda_2 - (1 - p_1) \lambda_1 D_1}{D_2} = \cdots = \frac{D_l \lambda_l - (1 - p_{l-1}) \lambda_{l-1} D_{l-1}}{D_l} = \cdots = \frac{D_k \lambda_k - (1 - p_{k-1}) \lambda_{k-1} D_{k-1}}{D_k}.$$

As a result

$$\frac{D_l \lambda_l - (1 - p_{l-1}) \lambda_{l-1} D_{l-1}}{D_l}$$

is independent of l. Similarly, if we keep l fixed and vary $\vec{i}$, we obtain that

$$\left[w q_1 \mu_1 i_1 R_{\vec{i}} + w \sum_{j=2}^{m} q_j \mu_j (i_j + 1) R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_1} + \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}} - R_{\vec{i}} \sum_{j=1}^{m} i_j \mu_j \right] \Big/ R_{\vec{i}}$$

is independent of $\vec{i}$. Therefore, there exists a constant x, which depends on w, but is independent of l and $\vec{i}$ such that

$$\frac{D_l \lambda_l - (1 - p_{l-1}) \lambda_{l-1} D_{l-1}}{D_l} = -x = \left[w q_1 \mu_1 i_1 R_{\vec{i}} + w \sum_{j=2}^{m} q_j \mu_j (i_j + 1) R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_1} + \sum_{j=1}^{m} (1 - q_j) \mu_j (i_j + 1) R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}} - R_{\vec{i}} \sum_{j=1}^{m} i_j \mu_j \right] \Big/ R_{\vec{i}}$$

and hence

$$D_l\lambda_l - (1-p_{l-1})\lambda_{l-1}D_{l-1} = -xD_l \quad l = 2, \ldots, k \tag{6}$$

$$wq_1\mu_1 i_1 R_{\vec{i}} + w\sum_{j=2}^{m} q_j\mu_j(i_j+1)R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_1}$$
$$+ \sum_{j=1}^{m}(1-q_j)\mu_j(i_j+1)R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_{j+1}}$$
$$- R_{\vec{i}}\sum_{j=1}^{m} i_j\mu_j = -xR_{\vec{i}}, \quad |\vec{i}| = s. \tag{7}$$

Solving (6) we find that

$$D_l = D_1\prod_{r=1}^{l-1}\frac{(1-p_r)\lambda_r}{x+\lambda_{r+1}} \quad l = 2, \ldots, k. \tag{8}$$

Substituting (8) to (3) and using (7) we find the relation between x and w

$$w = \sum_{l=1}^{k}\frac{p_l\lambda_l}{x+\lambda_l}\prod_{r=1}^{l-1}\frac{(1-p_r)\lambda_r}{x+\lambda_r} = f^*_{T_a}(x). \tag{9}$$

2.2. A Generating Function Technique

Equations (7) form a linear homogeneous system of $\binom{s+m-1}{s}$ equations with $\binom{s+m-1}{s}$ unknowns. Since the system is homogeneous, w must be chosen such that the determinant of the system is zero. Since $\binom{s+m-1}{s}$ is large, a direct determination of the values of w with this property is computationally unattractive. We, therefore, use a different method, and introduce the m variable generating function

$$U(\vec{z}) \triangleq \sum_{|\vec{i}|=s} R_{\vec{i}} z_1^{i_1}\ldots z_m^{i_m},$$
$$\vec{i} = (i_1, \ldots, i_m), \quad \vec{z} = (z_1, \ldots, z_m).$$

Multiplying (7) by $z_1^{i_1}\ldots z_m^{i_m}$ and summing for all $\vec{i}$, $|\vec{i}| = s$ we get the following partial differential equation of the first order

$$\sum_{j=1}^{m}\frac{\partial U(\vec{z})}{\partial z_j}(\mu_j z_j - wz_1 q_j\mu_j - (1-q_j)\mu_j z_{j+1})$$
$$= xU(\vec{z}). \tag{10}$$

For the derivation of (10) we use the identities

$$z_j\frac{\partial U(\vec{z})}{\partial z_j} = \sum_{|\vec{i}|=s} R_{\vec{i}} i_j z_1^{i_1}\ldots z_m^{i_m}$$
$$z_r\frac{\partial U(\vec{z})}{\partial z_j} = \sum_{|\vec{i}|=s} R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_r}(i_j+1)z_1^{i_1}\ldots z_m^{i_m},$$
$$r = 1, \quad j+1.$$

Our goal is to find the w satisfying the condition that the determinant of (7) is zero, so that $R_{\vec{i}} \neq 0$. But then $U(\vec{z})$ should be a nonzero polynomial in $\vec{z}$. Thus, the w satisfying (7) are the ones that allow $U(\vec{z})$ to be a nonzero polynomial. This last condition will lead to the determination of w in the next section.

2.3. The Method of Characteristics

Our goal is to solve (10) which is a linear partial differential equation (pde) of the first order in m variables. We will use a well known method, the method of characteristics, from the theory of pde's in order to solve it. We form the following system of ordinary differential equations.

$$\frac{dz_1}{(1-wq_1)\mu_1 z_1 - (1-q_1)\mu_1 z_2} = \ldots$$
$$= \frac{dz_j}{-wq_j\mu_j z_1 + \mu_j z_j - (1-q_j)\mu_j z_{j+1}} = \ldots$$
$$= \frac{dz_m}{-w\mu_m z_1 + \mu_m z_m} = \frac{dU(\vec{z})}{xU(\vec{z})} = dt \tag{11}$$

which can be written as

$$\frac{d\vec{z}}{dt} = A_m\vec{z} \tag{12}$$

where A_m is the $m \times m$ matrix

$$A_m = \begin{bmatrix} -wq_1\mu_1+\mu_1 & -(1-q_1)\mu_1 & 0 & \ldots & 0 \\ -wq_2\mu_2 & \mu_2 & -(1-q_2)\mu_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -wq_{m-1}\mu_{m-1} & 0 & \ldots & \mu_{m-1} & -(1-q_{m-1})\mu_{m-1} \\ -w\mu_m & 0 & \ldots & 0 & \mu_m \end{bmatrix}$$

To solve the system (12) we first find the eigenvalues of A_m using Proposition 1 below.

Proposition 1. *The eigenvalues of matrix A_m are the m roots $\theta_i(x)$ $i = 1, \ldots, m$ of the equation*

$$f^*_{T_a}(x)f^*_{T_s}(-\theta(x)) = 1. \tag{13}$$

Proof. We first prove that

$$|A_m - \theta I| = (1 - wf^*_{T_s}(-\theta))\prod_{j=1}^{m}(\mu_j - \theta).$$

Obviously

$$|A_1 - \theta I| = -wq_1\mu_1 + \mu_1 - \theta.$$

Noting that A_{r-1} is the principal minor of order $r-1$ of A_r, we expand the determinant $|A_r - \theta I|$ along the last column to find

$$|A_r - \theta I| = (\mu_r - \theta)|A_{r-1} - \theta I|$$
$$- wq_r\mu_r\sum_{j=1}^{r-1}(1-q_j)\mu_j.$$

We devide by $\prod_{j=1}^{r} (\mu_j - \theta)$ and take

$$d_r \triangleq \frac{|A_r - \theta I|}{\prod_{j=1}^{r} (\mu_j - \theta)} = d_{r-1} - \frac{wq_r\mu_r}{\mu_r - \theta} \prod_{j=1}^{r-1} \frac{(1 - q_j)\mu_j}{\mu_j - \theta}.$$

Solving this recurrence we find

$$d_m = d_1 - \sum_{r=2}^{m} \frac{wq_r\mu_r}{\mu_r - \theta} \prod_{j=1}^{r-1} \frac{(1 - q_j)\mu_j}{\mu_j - \theta}.$$

But since

$$d_1 = \frac{|A_1 - \theta I|}{\mu_1 - \theta} = 1 - w\frac{q_1\mu_1}{\mu_1 - \theta}$$

we find that

$$d_m = 1 - w \sum_{r=1}^{m} \frac{q_r\mu_r}{\mu_r - \theta} \prod_{j=1}^{r-1} \frac{(1 - q_j)\mu_j}{\mu_j - \theta}$$

$$= 1 - wf^*_{T_s}(-\theta).$$

Therefore

$$|A_m - \theta I| = (1 - wf^*_{T_s}(-\theta)) \prod_{j=1}^{m} (\mu_j - \theta).$$

The eigenvalues of A_m are the roots of the equation

$$(1 - wf^*_{T_s}(-\theta)) = 0$$

which combined with (9) gives (13).

Therefore, from (12) and (13) we find

$$z_i = \sum_{j=1}^{m} c_{i,j} e^{\theta_j(x)t} \quad i = 1, \ldots, m \tag{14}$$

where $\vec{C}_j \triangleq [c_{1,j}, \ldots, c_{m,j}]^T$ is the eigenvector of matrix A_m that corresponds to the eigenvalue $\theta_j(x)$. Also from (11)

$$U(\vec{z}(t)) = Ce^{tx}. \tag{15}$$

Since $\vec{C}_j$ is an eigenvector of A_m its components $c_{2,j}, \ldots, c_{m,j}$ are multiples of $c_{1,j}$ and thus

$$c_{i,j} = a_{i,j}c_{1,j}, \quad a_{1,j} \triangleq 1$$

where $a_{i,j}$ are found from $A_m\vec{C}_j = \theta_j(x)\vec{C}_j$ to be

$$a_{i,j} = w\sum_{r=i}^{m} \frac{q_r\mu_r}{\mu_r - \theta_j(x)} \prod_{l=i}^{r-1} \frac{(1 - q_l)\mu_l}{\mu_l - \theta_j(x)}.$$

From (15) we find that

$$e^t = \left[\frac{U(\vec{z})}{C}\right]^{1/x}$$

and thus

$$e^{\theta_j(x)t} = \left[\frac{U(\vec{z})}{C}\right]^{\theta_j(x)/x}.$$

As a result, (14) becomes

$$z_i = \sum_{j=1}^{m} b_j a_{i,j}[U(\vec{z})]^{\theta_j(x)/x} \quad i = 1, \ldots, m$$

where $b_j \triangleq c_{1,j}/C^{\theta_j(x)/x}$ (b_j are still undetermined).

We solve this system for the m coefficients b_j ($j = 1, \ldots, m$) using Cramer's rule and we find that

$$b_j = \frac{\begin{vmatrix} 1 & \ldots & 1 & z_1 & 1 & \ldots & 1 \\ a_{2,1} & \ldots & a_{2,j-1} & z_2 & a_{2,j-1} & \ldots & a_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & \ldots & a_{m,j-1} & z_m & a_{m,j+1} & \ldots & a_{m,m} \end{vmatrix}}{\begin{vmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ a_{2,1} & \ldots & a_{2,j-1} & a_{2,j} & a_{2,j+1} & \ldots & a_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & \ldots & a_{m,j-1} & a_{m,j} & a_{m,j+1} & \ldots & a_{m,m} \end{vmatrix}} \cdot [U(\vec{z})]^{-\theta_j(x)/x}. \tag{16}$$

This means

$$b_j = (b_{1,j}z_1 + \ldots + b_{m,j}z_m)[U(\vec{z})]^{-\theta_j(x)/x}, \qquad j = 1, 2, \ldots, m$$

where $b_{i,j}$ can be computed analytically from $a_{i,j}$ by expanding the subdeterminants in (16). As a result

$$[U(\vec{z})]^{\theta_j(x)/x} = \frac{1}{b_j}(b_{1,j}z_1 + \ldots + b_{m,j}z_m) \qquad (j = 1, \ldots, m). \tag{17}$$

We want to find a general solution of (10), which satisfies the condition that $U(\vec{z})$ is a multivariable integer polynomial of $z_1, \ldots, z_m$ of degree s.

Raising (17) to some integer i_j and multiplying these m equations we find

$$[U(\vec{z})]^{(\sum_{j=1}^{m} \theta_j(x)i_j)/x} = K \prod_{j=1}^{m} (b_{1,j}z_1 + \ldots + b_{m,j}z_m)^{i_j} \tag{18}$$

where $K \triangleq \prod_{j=1}^{m} 1/(b_j)^{i_j}$ is an undetermined constant, independent of $z_1, \ldots, z_m$.

Clearly (18) satisfies (10). For $U(\vec{z})$ to be a multivariable integer polynomial of $\vec{z}$ of degree s we demand

$$\frac{\sum_{j=1}^{m} \theta_j(x)i_j}{x} = 1 \quad \sum_{j=1}^{m} i_j = s \quad i_j \in Z^+. \tag{19}$$

Therefore, if the above conditions hold, we have found a solution of (10) that satisfies the *polynomiality* condition. Hence, the generating function $U(\vec{z})$ that corresponds to the combination $\vec{i} = (i_1, \ldots, i_m)$ is of the form

$$U_{\vec{i}}(\vec{z}) = K_{\vec{i}} \prod_{j=1}^{m} (b_{1,j}z_1 + \ldots + b_{m,j}z_m)^{i_j}. \qquad (20)$$

Let us summarize what we have shown up to this point. Our goal was to find the w satisfying the condition that the determinant of (7) is zero, so that the system of (7) has a nonzero solution. This last condition is, in turn, equivalent to the condition that the auxiliary generating function $U(\vec{z})$ is a nonzero multivariable polynomial. Solving for $U(\vec{z})$ and imposing the condition that $U(\vec{z})$ is a nonzero multivariable polynomial lead to (19), which are the equations that the x, and thus the $w = f^*_{T_a}(x)$, satisfy.

Thus in order to find w one proceeds as follows. Fix one of the $\binom{s+m-1}{s}$ vectors $\vec{i} = (i_1, \ldots, i_m)$ such that $\sum_{j=1}^{m} i_j = s$. First one solves the following equation for x.

$$\phi_{\vec{i}}(x) \triangleq i_1\theta_1(x) + i_2\theta_2(x) + \ldots + i_m\theta_m(x) = x,$$
$$i_1 + i_2 + \ldots + i_m = s. \qquad (21)$$

Here the $\theta_j(x)$ $(j = 1, \ldots, m)$ are the m roots of the polynomial equation of degree m

$$f^*_{T_a}(x) f^*_{T_s}(-\theta_j(x)) = 1. \qquad (22)$$

Now $w = f^*_{T_a}(x)$. As all $|w| > 1$ should be disregarded, we are only interested in the roots x that satisfy

$$|f^*_{T_a}(x)| < 1. \qquad (23)$$

In the next section, we prove that if $\rho < 1$ for each of the $\binom{s+m-1}{s}$ combinations of the vector $\vec{i}$, such that $|\vec{i}| = s$, there is exactly one root w inside the unit circle ($|w| < 1$). Therefore, there is a total of $\binom{s+m-1}{s}$ roots w and each is characterized by the system of equations (21), (22) and (23). Note that each root w satisfies a different equation. Thus, the use of the generating function enables us to completely characterize the equation that each root w satisfies.

2.4. The Basic Separability Theorem

In order to investigate the number of roots of (21) we prove the following theorem.

Theorem 2. *If* $\rho < 1$, *for every of the* $\binom{s+m-1}{s}$ *combinations of* $\vec{i}$, $|\vec{i}| = s$ (21) *has at least one root* x *that satisfies* (23).

Proof. First we prove that there are no roots x such that

$$\mathrm{Re}\{x\} < 0.$$

If $\mathrm{Re}\{x\} < 0$ then from (21) there exists a j $(1 \leq j \leq m)$ such that $\mathrm{Re}\{\theta_j(x)\} < 0$. Then

$$|f^*_{T_s}\{-\theta_j(x)\}| \triangleq \left| \int_0^\infty e^{\theta_j(x)t} f_{T_s}(t)\, dt \right|$$
$$\leq \int_0^\infty |e^{\theta_j(x)t}| f_{T_s}(t)\, dt$$
$$= \int_0^\infty e^{\mathrm{Re}\{\theta_j(x)\}t} f_{T_s}(t)\, dt < 1.$$

Therefore, combining the above strict inequality and (22), we conclude that since $f^*_{T_a}(x) f^*_{T_s}(-\theta_j(x)) = 1$, then $|f^*_{T_a}(x)| > 1$. Therefore, the assumption $\mathrm{Re}\{x\} < 0$ violates (23), and hence, there are no roots x such that $\mathrm{Re}\{x\} < 0$.

We will prove that for every combination $\vec{i}$ there exists a root x in the right half plane ($\mathrm{Re}\{x\} \geq 0$) which satisfies the system of equations (21) and (22) using the following fixed point theorem.

Schauder Fixed Point Theorem (see Hale 1980, p. 10). *Every continuous function* $f(x)$ *defined from a convex, bounded and closed region into itself has a fixed point* x_0, *i.e., there exists an* x_0 *such that* $f(x_0) = x_0$.

We will prove that there exists an M such that $\phi_{\vec{i}}(x)$ defined in (21) has a fixed point in the region $D_M \triangleq \{x: \mathrm{Re}\{x\} \geq 0, |x| \leq M\}$, in other words, that there exists a root x in the right half plane.

Proof. Clearly D_M is convex, bounded and closed (compact). Also, all functions $\theta_j(x)$ defined from (22) are continuous because they are roots of a polynomial equation, where all coefficients are continuous functions of x. Therefore, $\phi_{\vec{i}}(x)$ is a continuous function because it is a linear combination of continuous functions. In order to complete the proof that there is a fixed point it suffices to prove that there exists an M such that $\phi_{\vec{i}}(x)$ takes values in D_M. From (22), we have that for every $j = 1, \ldots, m$ $\mathrm{Re}\{\theta_j(x)\} \geq 0$ because if $\mathrm{Re}\{\theta_j(x)\} < 0$ then

$$|f^*_{T_s}\{-\theta_j(x)\}| \triangleq \left| \int_0^\infty e^{\theta_j(x)t} f_{T_s}(t)\, dt \right| < 1.$$

Also

$$|f^*_{T_a}(x)| \triangleq \left| \int_0^\infty e^{-xt} f_{T_a}(t)\, dt \right| \leq 1.$$

Then

$$|f^*_{T_a}(x) f^*_{T_s}\{-\theta_j(x)\}| < 1$$

and, therefore, (22) cannot possibly be satisfied. Thus $\mathrm{Re}\{\phi_{\vec{i}}(x)\} \geq 0$.

We can claim that there exists an M such that $|\theta_j(x)| \leq M_1 \triangleq M/s$. If not, for all M there exists a y such that $|\theta_j(y)| \geq M_1$, that is, $\theta_j(x)$ tends to infinity as $x \to y$. Since $\lim_{|x|\to\infty} f^*_{T_a}(x) = 0$, then from (22) in order for the product $f^*_{T_a}(x) f^*_{T_s}(-\theta_j(x))$ to be nonzero, $-\theta_j(x)$ must tend to a pole of $f^*_{T_s}(\cdot)$. Thus

$$\lim_{|x|\to\infty} \theta_j(x) = \mu, \quad j = 1, \ldots, m$$

which means that $\theta_j(x)$ is bounded at infinity. Therefore, y is finite and

$$\lim_{x\to y} \theta_j(x) = \infty$$

which contradicts the continuity of $\theta_j(x)$. Therefore, there exists an M such that $|\theta_j(x)| \leq M/s$ and therefore from (21) $|\phi_{\vec{i}}(x)| \leq M$, which proves that $\phi_{\vec{i}}(x)$ is from D_M into D_M, and thus, since D_M is convex and compact, has a fixed point in D_M. Therefore, we have shown that for every combination of $\vec{i}$ there exists a root of the system of (21) and (22). Furthermore, if $\mathrm{Re}\{x\} \geq 0$, $x \neq 0$ then $|f^*_{T_a}(x)| < 1$. Yet, the solution $x = 0$ is not excluded. In fact, for $\vec{i} = (0, \ldots, 0, s)$ $x = 0$ is a solution to (21) and (22). If there are two nonzero components i_k, i_l of the vector $\vec{i}$, we can easily check that $x = 0$ cannot be a solution to (21) and (22).

Therefore, in order to prove the theorem, we are led to the investigation of the roots for the m combinations of $\vec{i}$ where $m - 1$ components are 0 and one is equal to s. Using the following well known (see Volkovysky, Lunts and Aramovitch 1977, p. 107) implication of the *argument principle* from complex analysis, we prove that, when $\rho < 1$, then these roots are unique and nonzero: Let $\phi(z)$ be a meromorhic (analytic except poles) function in the domain G, which is analytic on its boundary C. If $|\phi(z)| < 1$ on C, then the number of roots of the equation $\phi(z) = 1$ in the domain G is equal to the number of poles of the function $\phi(z)$ in that domain.

Then, (21) becomes

$$h(x) \triangleq f^*_{T_a}(x) f^*_{T_s}\left(-\frac{x}{s}\right) = 1.$$

We will apply the above result in the domain E_1 of Figure 2. We prove that $|h(x)| < 1$ in the boundary of E_1. Then

1. $\mathrm{Re}\{x\} = 0 \quad (x = i\alpha \neq 0)$
   We easily get that $|f^*_{T_a}(x)| < 1$ and $|f^*_{T_s}(-x/s)| < 1$ from where
   $|h(x)| < 1.$

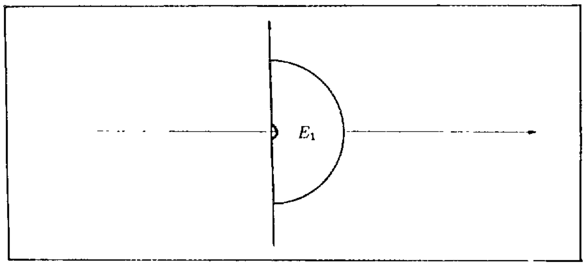

Figure 2. The domain E_1.

2. $\mathrm{Re}\{x\} > 0 \quad |x| = L \to \infty$
   Then $\lim_{|x|\to\infty} f^*_{T_a}(x) = 0$ and $\lim_{|x|\to\infty} f^*_{T_s}(-x/s) = 0$. Thus, for $|x| = L$ for some big enough L and $\mathrm{Re}\{x\} > 0$
   $|h(x)| < 1.$
3. $x \to 0^+$
   Using a Taylor expansion we find

$$f^*_{T_a}(x) = 1 - \frac{x}{\lambda} + o(x)$$

$$f^*_{T_s}\left(-\frac{x}{s}\right) = 1 + \frac{x}{s\mu} + o(x).$$

Thus

$$|h(x)| = \left|1 - x\left(\frac{1}{\lambda} - \frac{1}{s\mu}\right) + o(x)\right|$$

$$= \left|1 - \frac{x}{\lambda}(1-\rho) + o(x)\right| < 1 \quad \text{if } \rho < 1.$$

Since the function $h(x)$ is meromorhic in E_1, analytic in the boundary of E_1 and satisfies $|h(x)| < 1$ in the boundary, the number of roots of $h(x)$ in E_1 is equal to the number of poles of $h(x)$ in E_1, which is exactly m, because $f^*_{T_a}(x)$ does not have poles in E_1 and $f^*_{T_s}(-x/s)$ has exactly m poles. These m roots obviously satisfy (23) because $\mathrm{Re}\{x\} > 0$.

Since we have proved that there are no roots for $\mathrm{Re}\{x\} < 0$ we conclude that if $\rho < 1$ (21) has exactly m roots that satisfy (23), for the m combinations of $\vec{i}$ where $m - 1$ components are 0 and one is equal to s. Combining the result and the general proof that there exists a root for every combination of $\vec{i}$, we conclude that for every combination of the type $\vec{i} = (0, \ldots, 0, s, 0, \ldots, 0)$ there exists a unique root if $\rho < 1$. As a result, we conclude that if $\rho < 1$, for each of the $\binom{s+m-1}{s}$ combinations of $\vec{i}$, $|\vec{i}| = s$, (21) has at least one root x that satisfies (23).

Up to this point, we have established the existence of a root x that satisfies (21), (22) and (23) for every combination of $\vec{i}$. Furthermore, we have shown that for a particular type of combination $\vec{i}$ this root is unique, provided that $\rho < 1$. Under the condition that these roots are distinct and because there are $\binom{s+m-1}{s}$ combinations of $i_1, \ldots, i_m$, such that $\sum_{j=1}^{m} i_j = s$, we proved that there are at least $\binom{s+m-1}{s}$ roots of (21). This condition is clearly *almost always* satisfied, in the sense that the subset of distributions for which this condition does not hold has Lebesgue measure 0. However, we have not been able to construct any example in which this condition is violated. We conjecture that this condition will always be satisfied. In fact, we were able to prove this for the special case $m = 2$ (Bertsimas 1988). For $m = 1$ this condition holds from the well known G/M/s theory (see also special case 3 in Section 2.5).

We did not prove the uniqueness of the roots x in the general case using results of the present theory exclusively, but this is seen to hold by combining the result of Theorem 2 and the results of Pollaczek, who showed that the waiting time distribution for the G/C_m/s QS is a mixture of at most $\binom{s+m-1}{s}$ exponential terms, which implies that there are at most $\binom{s+m-1}{s}$ roots of (7). Bertsimas shows that if there are t roots of (7), then the waiting time distribution is a mixture of t exponential terms. Furthermore, de Smit (1983a) proved that under some conditions, which do not seem to have some probabilistic meaning, and using a matrix generalization of Rouché's theorem, if $\rho < 1$ there are $\binom{s+m-1}{s}$ roots for the G/H_m/s QS.

As a result of the above discussion, we conclude by combining our result of Theorem 2 and the results of Pollaczek and de Smit that there are exactly $\binom{s+m-1}{s}$ roots of (7), provided that $\rho < 1$. Furthermore, these roots satisfy (21) and (22). It is remarkable that the equations for the roots w depend only on the Laplace transforms of the interarrival and service time distributions.

In order to prove the uniqueness of the roots x for every combination of $\vec{i}$, using the results of the present theory exclusively, one might use Rouché's theorem, but the problem arises that the functions $\theta_j(x)$ defined from (22) might not be analytic. In Section 2.5, we examine some special cases in which we were able to prove the uniqueness of the roots w using the present theory exclusively.

Remarks

1. Since we proved that there are $\binom{s+m-1}{s}$ roots w that correspond to the $\binom{s+m-1}{s}$ combinations of $\vec{i} = (i_1, \ldots, i_m)$ such that $\sum_{j=1}^{m} i_j = s$ we label these roots w_j ($j = 1, \ldots, \binom{s+m-1}{s}$). We denote by $D_{l,j}$, $R_{\vec{i},j}$ the coefficients corresponding to w_j. Also, $U_j(\vec{z})$ denotes the generating function corresponding to w_j. Then from (20) $U_j(\vec{z})$ is given by

$$U_{j}(\vec{z}) = K_j \prod_{r=1}^{m} (b_{1,r}(w_j)z_1 + \ldots + b_{m,r}(w_j)z_m)^{i_r}. \quad (24)$$

From the definition of

$$U_j(\vec{z}) \triangleq \sum_{|\vec{i}|=s} R_{\vec{i},j} z_1^{i_1} \ldots z_m^{i_m}$$

the coefficient of z_m^s is equal to $R_{(0,\ldots,s),j}$. From (24) the coefficient of z_m^s is equal to

$$K_j b_{m,1}^{i_1}(w_j) \ldots b_{m,m}^{i_m}(w_j).$$

Thus

$$U_j(\vec{z}) = R_{(0,\ldots,s),j} \prod_{r=1}^{m} \cdot \left(\frac{b_{1,r}(w_j)z_1 + \ldots + b_{m,r}(w_j)z_m}{b_{m,r}(w_j)}\right)^{i_r}. \quad (25)$$

2. This above analysis explains the title of Section 2.4. We proved that there are $\binom{s+m-1}{s}$ roots w_j, each of which satisfies a different equation, corresponding to the $\binom{s+m-1}{s}$ combinations of $\vec{i}$, such that $|\vec{i}| = s$. This separability property of the equations from which the roots w_j can be computed is theoretically interesting because the equations for the roots w_j involve only the Laplace transforms of the interarrival and service time distributions, but is also computationally useful.

2.5. Some Special Cases

$C_k/C_m/1$. Since the only combinations of $\vec{i}$ for $s = 1$ are of the type $\vec{i} = (0, \ldots, 1, \ldots, 0)$ we have already proved that there are exactly $a(1, m) = m$ roots if $\rho < 1$.

If we permit complex transition rates (λ_i, μ_i) the proof is still valid, but the poles of $f_{T_s}^*(-x)$ are not necessarily on the real line anymore (see also Remark 3 in Section 2.6). For computational purposes, it is interesting to investigate when these m roots are real or complex. We assume first that the m poles of the service time distribution are distinct and there are no zeros of $f_{T_a}^*(x)$ that coincide with any pole of $f_{T_s}^*(-x)$.

If $g(x) \triangleq f_{T_a}^*(x)f_{T_s}^*(-x) - 1$, then $\lim_{x\to 0^+} g(x) = -x(1 - \rho)/\lambda < 0$ for $x > 0$ and $\lim_{x\to\infty} g(x) = -1$. Graphically $g(x)$ is presented in Figure 3. We observe in the figure that all the m roots are real. If there are zeros of $f_{T_a}^*(x)$ that coincide with poles of $f_{T_a}^*(-x)$, then there may be roots that are complex.

If the poles of $f_{T_s}^*(-x)$ are not distinct, then there may be roots that are complex. Consider for example

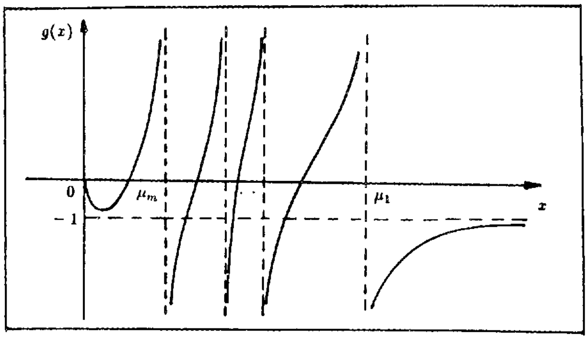

Figure 3. $g(x)$ with all m poles of $f^*_{T_s}(-x)$ real.

the case of $C_k/E_m/1$. In this case, $g(x)$ is presented in Figure 4. If m is odd then there is only one real root (Figure 4). If m is even then there are 2 real roots. As a result, since the algorithmic complexity of the determination of the roots increases if the roots are complex, we can say that the algorithmic complexity increases when the service time distribution becomes more homogeneous (E_m, for example, in the sense that the rates of the stages are the same).

$E_k/E_m/s$. In this case (22) becomes

$$\left(\frac{k\lambda}{k\lambda + x}\right)^k\left(\frac{m\mu}{m\mu - \theta_j(x)}\right)^m = 1$$

$$\Rightarrow \theta_j(x) = m\mu\left(1 - \exp\left(\frac{2\pi(j-1)i}{m}\right)\left(\frac{k\lambda}{k\lambda + x}\right)^{k/m}\right)$$

$$j = 1, \ldots, m. \quad (26)$$

Substituting (26) to (21) we get

$$sm\mu - m\mu\left(\frac{k\lambda}{k\lambda + x}\right)^{k/m}\sum_{j=1}^{m} i_j\exp\left(\frac{2\pi(j-1)i}{m}\right) = x.$$

If we define

$$\varepsilon_{\vec{i}} \triangleq \sum_{j=1}^{m} i_j\exp\left(\frac{2\pi(j-1)i}{m}\right) \quad (|\varepsilon_{\vec{i}}| < s)$$

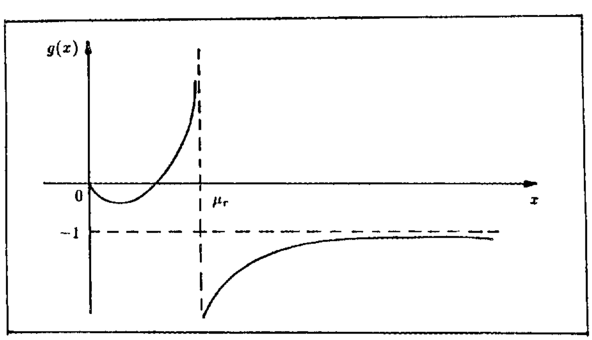

Figure 4. $g(x)$ with only 1 pole of $f^*_{T_s}(-x)$ real.

then we must solve

$$x - sm\mu + m\mu\left(\frac{k\lambda}{k\lambda + x}\right)^{k/m}\varepsilon_{\vec{i}} = 0.$$

We apply Rouché's theorem in the domain E_1 of Figure 2. Then for $|x| = L \to \infty$

$$\left|m\mu\left(\frac{k\lambda}{k\lambda + x}\right)^{k/m}\varepsilon_{\vec{i}}\right| < sm\mu < |x - sm\mu|.$$

In particular, for $x = 0$ and $\vec{i} = (s, 0, \ldots, 0), \ldots, \vec{i} = (0, \ldots, 0, s)$, the above strict inequality becomes equality. Thus, in order to have the required strict inequality so that we can apply Rouché's theorem, we consider a small semicircle and use a Taylor expansion to take as $x \to 0^+$ with $\mathrm{Re}\{x\} > 0$

$$\left|m\mu\left(\frac{k\lambda}{k\lambda + x}\right)^{k/m}\varepsilon_{\vec{i}}\right|$$

$$= \left|sm\mu\left(1 - \frac{x}{\lambda m} + o(x)\right)\right| < |x - sm\mu|$$

$$\text{if } \rho < 1.$$

In the boundary $x = i\alpha \neq 0$ one can verify after straightforward algebraic manipulations that the above strict inequality holds.

Thus, if $\rho < 1$, we conclude from Rouché's theorem that for each $\varepsilon_{\vec{i}}$ there is a unique root in E_1, which satisfies (23). Furthermore, since we have proved that there are no roots for $\mathrm{Re}\{x\} < 0$ and the radius L of the domain E_1 can get arbitrarily large we conclude that if $\rho < 1$ there is a unique root for every combination of $\vec{i}$ that satisfies (23).

$C_k/M/s$. For $m = 1$ we find that (22) becomes

$$f^*_{T_a}(x)f^*_{T_s}\left(-\frac{x}{s}\right) = 1 \Rightarrow x = s\mu(1 - f^*_{T_a}(x)).$$

Then w is the unique real root of the equation

$$w = f^*_{T_a}(s\mu(1 - w))$$

which is the well known result from G/M/s theory. At this point, we remark that our result which was obtained for the class of C_k interarrival distributions is still valid for a general distribution. This observation comes to support the conjecture in Section 3 that our solution is still valid even for the G/R/s QS.

$E_k/C_2/s$. This QS, which was studied by Bertsimas and Papaconstantinou (1988), has the very attractive property that all the $a(s, 2) = s + 1$ roots are real and, thus, the algorithmic complexity of this QS is low. In fact, an $O(s^3)$ real arithmetic algorithm was proposed.

2.6. General Remarks

1. We will investigate under which conditions we can find an explicit equation for x, in the sense that (21) is an implicit equation involving the functions $\theta_j(x)$ $(j = 1, \ldots, m)$, which are not known explicitly. This property is algorithmically useful because an explicit equation for x can be solved easily by numerical means. We exploit the fundamental result in the theory of polynomial equations. Since (21) is a polynomial equation for $\theta(x)$ (which has m roots, $\theta_1(x), \ldots, \theta_m(x)$), a closed-form expression for $\theta_j(x)$ can only be found for $m \leq 4$. For $m \geq 5$ we cannot, in general, find an explicit formula. For example, for $m = 2$ the $s + 1$ roots $x(w_j)$ $(w_j = f^*_{T_a}(x(w_j)))$ satisfy

$$(s - 2j)\sqrt{\Delta(f^*_{T_a}(x(w_j))} - s(\mu_1 + \mu_2) + q_1 s\mu_1 f^*_{T_a}(x(w_j)) + 2x(w_j) = 0, \quad j = 0, \ldots, s$$

where

$$\Delta(y) \triangleq (\mu_2 - \mu_1 + yq_1\mu_1)^2 + 4\mu_1\mu_2(1 - q_1)y.$$

2. The complexity of the problem increases extremely fast with the number of roots, which increase exponentially in s, m, when both s, m vary. For this reason, it is better to use low values for m ($m = 2$, 3, 4) to approximate a service time pdf that avoids the venture of determining exponentially many roots. In the opinion of the author, in a practical situation the value of $m = 2$ is a tradeoff between accuracy and simplicity, which has the additional advantage of the rather unexpected real arithmetic.
3. In the attempt to investigate Cox's idea of introducing complex transition rates, to obtain complete generality in synthesizing any pdf with rational Laplace transform, we observe from the general proof that there are $\binom{s+m-1}{s}$ roots of (21) that did not depend on the assumption that $\mu_1, \ldots, \mu_m$ or $\lambda_1, \ldots, \lambda_k$ are real. Since, in order to have a valid pdf with complex poles we need $m > 2$, the simplest multiserver model involving complex transition rates is $C_3/C_3/s$, which has $[(s + 2)(s + 1)]/2 = O(s^2)$ complex roots.
4. A promising idea lies in the exploitation of an old and widely used idea in the field of Electrical Engineering, namely transfer functions. By reducing the order of a transfer function, which corresponds in queueing theory terms to the rational Laplace transform of the pdfs, we can find an excellent approximation of a large order pdf by a low order pdf. This decreases tremendously the algorithmic complexity of the problem.

2.7. The Algorithm for the Unsaturated Probabilities

Returning to the assumed form of the probabilities $P_{n,l,\vec{i}}$, $n \geq s$, $l = 1, \ldots, k$, $|\vec{i}| = s$ we observe that the most general solution under the condition that the roots w_j are distinct must be

$$P_{n,l,\vec{i}} = \sum_{j=1}^{a(s,m)} D_{l,j} R_{\vec{i},j} w_j^n \qquad n \geq s, \quad l = 1, \ldots, k, \quad |\vec{i}| = s \tag{27}$$

where from (8)

$$D_{l,j} = D_{1,j} \prod_{r=1}^{l-1} \frac{(1 - p_r)\lambda_r}{x(w_j) + \lambda_{r+1}} \qquad l = 2, \ldots, k$$

and $R_{\vec{i},j}$ satisfy (7). For each j corresponding to the root w_j the coefficients $R_{\vec{i},j}$ satisfy a system of $a(s, m) = \binom{s+m-1}{s}$ linear homogeneous equations (7). Thus, for a fixed j we can find the ratios $R_{\vec{i},j}/R_{(0,\ldots,0,s),j} \triangleq f(\vec{i}, w_j)$ recursively from (7). In order to determine the saturated probabilities only the coefficients $B_j \triangleq D_{1,j}R_{(0,\ldots,0,s),j}$ remain unknown. Therefore

$$P_{n,l,\vec{i}} = \sum_{j=1}^{a(s,m)} B_j \left(\prod_{r=1}^{l-1} \frac{(1 - p_r)\lambda_r}{x(w_j) + \lambda_{r+1}}\right) f(\vec{i}, w_j) w_j^n \qquad n \geq s, \quad l = 1, \ldots, k, \quad |\vec{i}| = s.$$

Furthermore, from (25) we observe that the generating function of $f(\vec{i}, w_j)$ is

$$G_j(\vec{z}) = \frac{U_j(\vec{z})}{R_{(0,\ldots,s)j}} = \sum_{|\vec{i}|=s} f(\vec{i}, w_j) z_1^{i_1} \ldots z_m^{i_m} = \prod_{r=1}^{m} \left(\frac{b_{1,r}(w_j)z_1 + \ldots + b_{m,r}(w_j)z_m}{b_{m,r}(w_j)}\right)^{i_r}. \tag{28}$$

Up to this point the only remaining unknowns are the coefficients B_j and the unsaturated probabilities $P_{n,l,\vec{i}}$, $n < s$, $l = 1, \ldots, k$, $|\vec{i}| = n$. Thus, we have reduced our problem to one with a finite number of unknowns. There are two strategies for finding these unknowns.

Strategy A

1. Using (2a) for $n < s$ we express the unsaturated probabilities $P_{n,l,\vec{i}}$ as linear combinations of the coefficients of B_j, that is, finding recursively from (1a) and (2a) the coefficients $g(n, l, \vec{i}, j)$ in the expansion

$$P_{n,l,\vec{i}} = \sum_{j=1}^{a(s,m)} B_j g(n, l, \vec{i}, j) \qquad n < s, \quad l = 1, \ldots, k, \quad |\vec{i}| = n. \tag{29}$$

Thus after this step only the coefficients B_j $(j = 1, \ldots, a(s, m))$ remain unknown.

2. Using the identities $P_{n-1,k,\vec{i}} = 0 \; n < s, \; |\vec{i}| = n$ we find

$$\sum_{n=0}^{s} \binom{n+m-1}{n} = \binom{s+m}{s}$$

linear homogeneous equations for B_j. Selecting $\binom{s+m-1}{s} - 1$ of them and using the normalization equation

$$\sum_{n,l,\vec{i}} P_{n,l,\vec{i}} = 1 \tag{30}$$

we find a linear nonhomogeneous system of $a(s, m)$ equations with the $a(s, m)$ unknowns B_j.

Strategy B

There are $k \sum_{n=0}^{s-1} \binom{n+m-1}{n} = k\binom{s+m-1}{s-1}$ unsaturated probabilities $P_{n,l,\vec{i}}$ and $\binom{s+m-1}{s}$ unknown coefficients B_j. Using (1a) and (2a) for $n < s$ we find $k\binom{s+m-1}{s-1}$ equations for $P_{n,l,\vec{i}}$. Also, the equations (1) for $l = 1$ and $n = s$ give another $\binom{s+m-1}{s}$ equations that involve the unknown quantities $P_{n,l,\vec{i}}$ $(n < s)$ and B_j. So, we have a linear homogeneous system of $k\binom{s+m-1}{s-1} + \binom{s+m-1}{s}$ equations with the same number of unknowns. Using the normalization equation (30) we find a linear nonhomogeneous system, which can be solved by numerical methods.

3. THE SYSTEM-SIZE PROBABILITY DISTRIBUTIONS AND THE USUAL PERFORMANCE MEASURES

In this section, we find closed-form expressions for the quantities P_n, $P^-_{n,\vec{i}}$, P^-_n, P^+_n (see Section 1.1 for the definition of these quantities) for $n \geq s$. Also closed-form expressions are provided in Section 3.2 for the usual performance measures.

3.1. System-Size Probability Distributions

Concerning the distribution of N we state the following proposition.

Proposition 3. *The general-time probabilities of the number of customers in the system have the form*

$$P_n = \begin{cases} \displaystyle\sum_{j=1}^{a(s,m)} B_j G_j(\vec{1}) \frac{\lambda_1 + x(w_j)}{x(w_j)} (1 - w_j) w_j^n & \text{if } n \geq s \\ \displaystyle\sum_{j=1}^{a(s,m)} B_j \sum_{l=1}^{k} \sum_{|\vec{i}|=n} g(n, l, \vec{i}, j) & \text{if } n < s \end{cases} \tag{31}$$

where $G_j(\vec{1}) = \sum_{|\vec{i}|=s} f(\vec{i}, w_j)$.

Proof. In general

$$P_n = \sum_{l=1}^{k} \sum_{|\vec{i}|=\min(n,s)} P_{n,l,\vec{i}}.$$

Then for $n \geq s$, using (27), we take

$$P_n = \sum_{j=1}^{a(s,m)} \left(\sum_{l=1}^{k} D_{l,j}\right)\left(\sum_{|\vec{i}|=s} R_{\vec{i},j}\right) w_j^n.$$

But $\sum_{|\vec{i}|=s} R_{\vec{i},j} = U_j(\vec{1}) = G_j(\vec{1}) R_{(0,\ldots,s),j}$ from the definition of the generating function $U_j(\vec{z})$ and by using (28). Also from (8)

$$\sum_{l=1}^{k} D_{l,j} = D_{1,j} \sum_{l=1}^{k} \prod_{r=1}^{l-1} \frac{(1 - p_r)\lambda_r}{x(w_j) + \lambda_{r+1}}$$

$$= D_{1,j} \frac{\lambda_1 + x(w_j)}{x(w_j)} (1 - w_j)$$

where we have used the identity

$$w_j = f^*_{T_a}(x(w_j)) = \sum_{l=1}^{k} \frac{p_l \lambda_l}{x(w_j) + \lambda_l} \prod_{r=1}^{l-1} \frac{(1 - p_r)\lambda_r}{x(w_j) + \lambda_r}.$$

Therefore

$$P_n = \sum_{j=1}^{a(s,m)} D_{1,j} \frac{\lambda_1 + x(w_j)}{x(w_j)} \cdot U_j(\vec{1}) w_j^n (1 - w_j) \quad n \geq s. \tag{32}$$

Using the definition of $B_j = D_{1,j} R_{(0,\ldots,0,s),j}$ (31) follows for $n \geq s$. For $n < s$, using (29), (31) follows easily.

Conjecture

Although the method of stages we present is not immediately extendable to distributions, which do not have rational Laplace transform, we believe that this separability property holds for the more general model GI/R/s, but does not hold for GI/G/s where G for the service time pdf does not belong to the class R. The reason for this difference is that it is the structure of class R and its probabilistic interpretation that enable us to *separate* the equations for the roots w_j. Summarizing, we conjecture that for the GI/R_m/s QS

$$P_n = \sum_{j=1}^{s+m-1} L_j w_j^n \quad n \geq s$$

where w_j are the $\binom{s+m-1}{s}$ roots of the system of equations

$$w = f^*_{T_a}(x) \quad (|w| < 1)$$

$$\sum_{j=1}^{m} i_j \theta_j(x) = x \quad \text{such that} \quad \sum_{j=1}^{m} i_j = s$$

$$f^*_{T_a}(x) f^*_{T_s}(-\theta_i(x)) = 1 \quad (i = 1, \ldots, m).$$

Concerning the prearrival probabilities we prove the following proposition.

Proposition 4. *The prearrival probabilities $P^-_{n,\vec{i}}$, P^-_n and the post-departure probabilities P^+_n for $n \geq s$ can be expressed as*

$$P^-_{n,\vec{i}}=\frac{1}{\lambda}\sum_{j=1}^{a(s,m)} B_j f(\vec{i}, w_j)(\lambda_1+x(w_j))w_j^{n+1} \qquad n \geq s, \quad |\vec{i}| = s \quad (33)$$

$$P^-_n = P^+_n = \frac{1}{\lambda}\sum_{j=1}^{a(s,m)} B_j G_j(\vec{1}) \cdot (\lambda_1 + x(w_j))w_j^{n+1} \quad n \geq s. \quad (34)$$

Proof. If we define the event AAO $\triangleq$ arrival about to occur in $(t, t+\delta t)$ then we take

$$P^-_{n,\vec{i}}=\Pr\{N=n,\vec{R}=\vec{i}\mid \text{AAO}\}$$
$$=\frac{\Pr\{N=n\cap\vec{R}=\vec{i}\cap \text{AAO}\}}{\Pr\{\text{AAO}\}}$$
$$=\frac{\Pr\{\cup_{l=1}^{k}(N=n\cap\vec{R}=\vec{i}\cap R_a=l)\cap \text{AAO}\}}{\Pr\{\cup_{l=1}^{k}R_a=l\cap \text{AAO}\}}$$
$$=\frac{\sum_{l=1}^{k}\Pr\{\text{AAO}\mid N=n\cap\vec{R}=\vec{i}\cap R_a=l\}\cdot\Pr\{N=n\cap\vec{R}=\vec{i}\cap R_a=l\}}{\sum_{l=1}^{k}\Pr\{\text{AAO}\mid R_a=l\}\Pr\{R_a=l\}}$$

But since

$$\Pr\{\text{AAO}\mid N=n\cap\vec{R}=\vec{i}\cap R_a=l\} = \Pr\{\text{AAO}\mid R_a=l\} = \lambda_l p_l \delta t$$

and

$$\Pr\{R_a=l\} = \frac{(1/\lambda_l)\sum_{r=l}^{k}\{p_r\prod_{m=1}^{r-1}(1-p_m)\}}{1/\lambda}$$

we take

$$P^-_{n,\vec{i}}=\frac{\sum_{l=1}^{k}\lambda_l p_l P_{n,l,\vec{i}}}{\sum_{l=1}^{k}\lambda_l p_l(\lambda/\lambda_l)\sum_{r=l}^{k}\{p_r\prod_{m=1}^{r-1}(1-p_m)\}}$$
$$=\frac{1}{\lambda}\sum_{l=1}^{k}\lambda_l p_l P_{n,l,\vec{i}} \quad (35)$$

since $\sum_{l=1}^{k} p_l \sum_{r=1}^{k}\{p_r\prod_{m=1}^{r-1}(1-p_m)\} = 1$.

Therefore, using (27) and (35) we have

$$P^-_{n,\vec{i}} = \frac{1}{\lambda}\sum_{j=1}^{a(s,m)}\left\{\sum_{l=1}^{k}\lambda_l p_l D_{l,j}\right\} R_{\vec{i},j} w_j^n. \quad (36)$$

Also from (8)

$$\sum_{l=1}^{k}\lambda_l p_l D_{l,j} = \sum_{l=1}^{k}\lambda_l p_l D_{1,j}\prod_{r=1}^{l-1}\frac{(1-p_r)\lambda_r}{x(w_j)+\lambda_{r+1}}$$
$$= D_{1,j}(\lambda_1 + x(w_j))w_j.$$

Thus from (36), (33) follows. Also

$$P^-_n = \sum_{|\vec{i}|=s} P^-_{n,\vec{i}}$$
$$=\frac{1}{\lambda}\sum_{j=1}^{a(s,m)} B_j(\lambda_1 + x(w_j))w_j^{n+1}\sum_{|\vec{i}|=s} f(\vec{i}, w_j).$$

From $\sum_{|\vec{i}|=s} f(\vec{i}, w_j) = G_j(\vec{1})$ and the general relation $P^-_n = P^+_n$ which holds for the GI/G/s QS, (34) follows.

3.2. System Performance Measures

Mean queue length

If L_q is the length of the queue then

$$E\{L_q\} = \sum_{n=s}^{\infty}(n-s)P_n. \quad (37)$$

We substitute (31) into (37) and find

$$E\{L_q\} = \sum_{j=1}^{a(s,m)} B_j G_j(\vec{1})\frac{\lambda_1 + x(w_j)}{x(w_j)}\cdot\frac{w_j^{s+1}}{1-w_j}. \quad (38)$$

Proportion of time all servers are busy, P_{busy}

$$P_{\text{busy}} = \sum_{n=s}^{\infty} P_n$$
$$= \sum_{j=1}^{a(s,m)} B_j G_j(\vec{1})\frac{\lambda_1 + x(w_j)}{x(w_j)} w_j^s. \quad (39)$$

Probability of nonzero waiting time

$$\Pr\{T_q > 0\}$$
$$=\sum_{n=s}^{\infty} P^-_n = \frac{1}{\lambda}\sum_{j=1}^{a(s,m)} B_j G_j(\vec{1})(\lambda_1 + x(w_j))\frac{w_j^{s+1}}{1-w_j}. \quad (40)$$

4. COMPUTATIONAL AND COMPLEXITY CONSIDERATIONS

Strongly critical remarks have been made in the past both on the merits of queueing theory (see e.g., Neuts, p. 42) and on unwarranted algorithmic claims in the applied literature on stochastic models. Being fully aware of the need for the probabilist to be closely involved with the algorithmic analysis of a problem, we have undertaken the task of programming and testing our algorithms.

4.1. The Algorithm

In order to extract numerical results from the formulas presented in Sections 2 and 3 we propose the following algorithm based on Strategy A (see Section 2.7).

1. Determination of $\binom{s+m-1}{s}$ roots w_j of the system of (21), (22), (23), and (9).
2. Determination of the coefficients $f(\vec{i}, w_j)$ from (7).

Table I
Computational Requirements for the Solution of the $C_k/C_m/s$

Step of Algorithm	Time Requirement	Memory Requirement
(1)	$O(a(s,m))$	$O(a(s,m))$
(2)	$O(a^2(s,m))$	$O(a^2(s,m))$
(3)	$O(k^3\,a^3(s,m))$	$O(ka^2(s,m))$
(4)	$O(a^3(s,m))$	$O(a^2(s,m))$

3. Determination of $P_{n,l,i}$ for $n < s$ as linear combinations of B_j.
4. Determination of the $\binom{s+m-1}{s}$ unknowns B_j as a solution of a linear system with $\binom{s+m-1}{s}$ equations.

4.2. Complexity Considerations

In Table I we show the computational complexity of each step of the algorithm with respect to memory and time requirements. From this table we can make the following observations. Since from a computational point of view the *heaviest* part of the algorithm is the third step, the time complexity of this algorithm is $O(k^3a^3(s, m))$, which for fixed m, is polynomial in the number of servers s and for fixed s, is polynomial in m. The algorithm is exponential if both s and m vary, but it is always polynomial in k. These results verify that the complexity of the analysis of the $C_k/C_m/s$ QS increases much faster with the service time than with the interarrival time pdf. That is, we expect that the derivation of numerical results for the $M/C_m/s$ QS is much harder than for the $C_m/M/s$ QS, for example.

4.3. The Numerical Solution of the $C_2/C_2/s$ QS

To fully gauge the performance of the proposed algorithm we programmed it in FORTRAN on a SUN 3. The reasons we selected this model are

1. it is representative of the general behavior of the algorithm for more general models;
2. it is in real arithmetic;
3. its complexity $O(s^3)$ is not very high;
4. this model allows the determination of exact results when the coefficients of variation of the interarrival and of the service time pdf are both greater than ½.

Merely as an illustration of the stability and accuracy of the present algorithm, Table II presents a few typical results for the $C_2/C_2/15$ case with $\rho = 0.9$ and the conditions $\mu_1 = 2\mu$, $q_1 = 1 - 1/V_s^2$, $\mu_2 = \mu_1(1 - q_1)$, $\lambda_1 = 2\lambda$, $p_1 = 1 - 1/V_a^2$, $\lambda_2 = \lambda_1(1 - p_1)$ (two moment-fit; V_a^2, V_s^2 are the coefficients of variation of the interarrival and service distributions, respectively).

4.4. The Numerical Solution of the $E_k/C_2/s$ QS

This QS is solved by Bertsimas and Papaconstantinou. In order to illustrate the dependence and the sensitivity of this algorithm on k we prepared another program in FORTRAN on a CYBER 171 for the analysis of the $E_k/C_2/s$ QS.

After careful tests of our computer programs we produced extensive numerical results for a wide range of the parameters s, ρ, k, μ_1, μ_2 and q which are in agreement with the results of Groenevelt et al. for the $M/H_2/s$ and $M/E_{1,2}/s$ and of de Smit (1983b) for the $M/H_2/s$, $E_2/H_2/s$ and $E_5/H_2/s$ systems. In particular, the algorithm was tested for values of V_s^2 up to 100, for values of k up to 50 and for values of s up to 40. In all cases, the algorithm produced reliable results as it satisfied all the internal accuracy checks.

In Table III we illustrate the dependence of the mean waiting time $\mu E\{T_q\}$ (in units of mean service

Table III
$\mu E\{T_q\}$ for the $E_k/C_2/15$ QS as a Function of $V_a^2 = 1/k$, V_s^2

	V_s^2				
κ	0.5	0.8	2.0	5.0	10.0
1	0.3067	0.3647	0.5784	1 0810	1.8902
5	0.1124	0.1661	0.3685	0.8558	1.6507
10	0.0909	0.1431	0.3431	0.8279	1.6207
15	0.0839	0.1356	0.3347	0.8186	1.6107
20	0.0805	0.1319	0.3305	0.8140	1.6057

Table II
$\mu E\{T_q\}$ as a Function of V_a^2, V_s^2 for the $C_2/C_2/15$ QS

	V_s^2					
V_a^2	0.5	0.8	2.0	5.0	10.0	50.0
0.5	0.1816	0.2375	0.4489	0.9498	1.7542	7.9025
0.8	0.2561	0.3133	0.5283	1.0334	1.8416	8.0838
2.0	0.5677	0.6259	0.8472	1.3662	2.1889	8.5139
5.0	1.3939	1.4490	1.6620	2.1953	3.0519	9.5035
10.0	2.8251	2.8676	3.0575	3.5764	4.4513	11.0235
50.0	14.5984	14.6074	14.6636	14.9542	15.6819	22.7259

time) on k for the model $E_k/C_2/15$ ($\rho = 0.9$, $\mu_1 = \mu_2$ for $V_s^2 < 1$ and $q_1 = \mu_1/(\mu_1 + \mu_2)$ for $V_s^2 > 1$). This selection of parameters coincides with that of Groenevelt et al. but differs from de Smit's (1983b). Observe that the numerical results as k increases converge quickly to the corresponding limiting (for $k \to \infty$) $D/C_2/s$ and thus render any computations for k greater than a certain small value almost unnecessary.

Finally, in Figure 5 we present the computational times in CPU seconds on a CYBER 171 as functions of k and s.

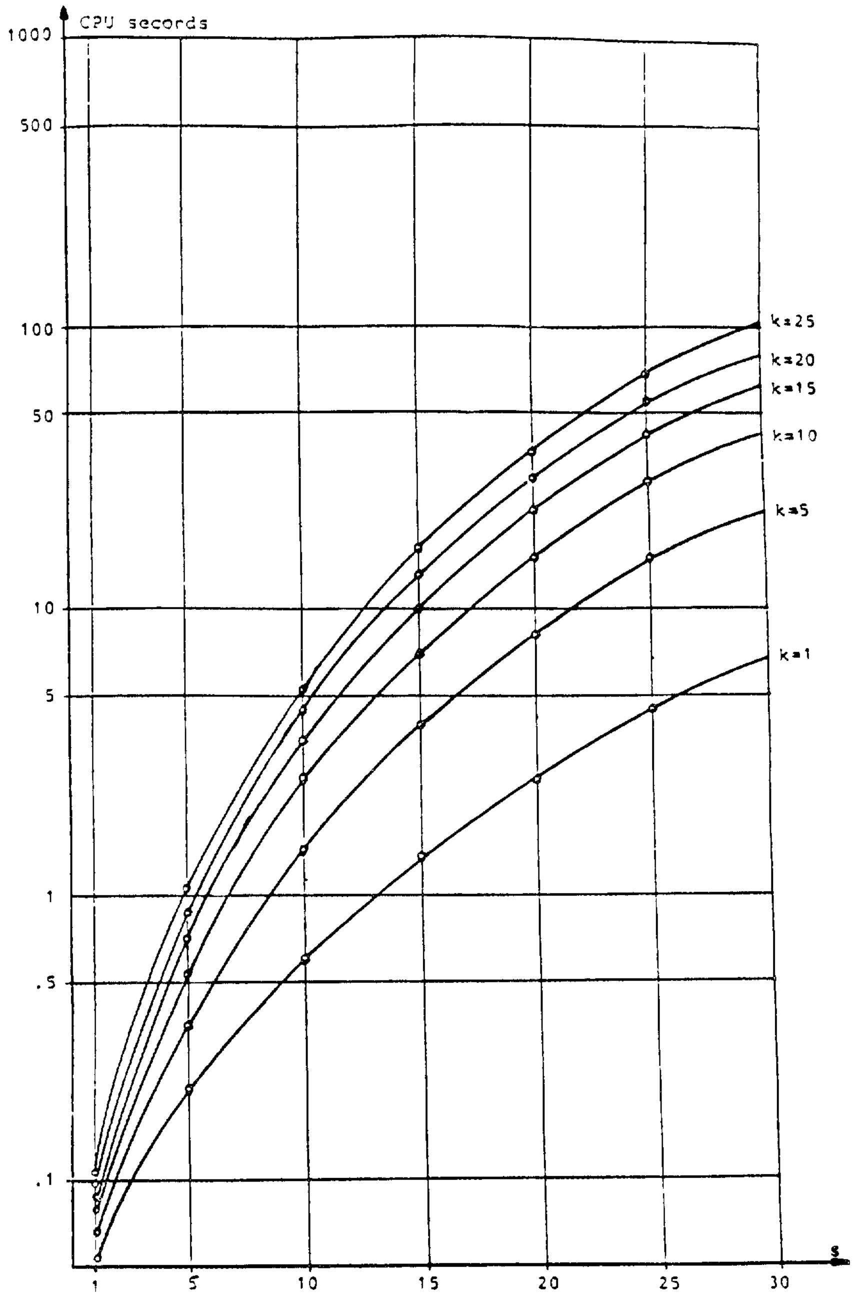

Figure 5. Computational times in CPU seconds for the $E_k/C_2/s$ QS on a CYBER 171.

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